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On the Scattering Theorems of Pearson and Ismagilov

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Pearson's trace theorem is carried over to the case of unbounded identification operator and the extended version is used to prove a generalization of Ismgilove multichannel scattering theorem.

1. INTRODUCTION

Given self-adjoint (not necessarily bounded) linear operators H_1 and H_2 in a separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and a bounded linear operator $J: \mathscr{H}_1 \to \mathscr{H}_2$, $J\mathscr{D}(H_1) \subset \mathscr{D}(H_2)$; if the closure of H_2J–JH_1 is of trace class, then the strong limits $\Omega_{\pm}(H_2, H_1; J) =$ $s\text{-lim}_{t \to \infty} e^{itH_2}Je^{-itH_1}P_a(H_1)$ exist on the entire Hilbert space, $P_a(H_1)$ being the orthogonal projection on the space of absolute continuity of H_1. This is the trace theorem of Pearson [4] that generalizes a theorem due to Belopol'skii and Birman [1] which itself is an extension of Kato–Rosenblum scattering theorem [3, Theorem X.4.4]. In this note, the following unbounded version of Pearson's theorem is obtained.

THEOREM 1. *Let H_1 and H_2 be self-adjoint in separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Let $J: \mathscr{D}(J) \subset \mathscr{H}_1 \to \mathscr{H}_2$ be a densely defined closed (not necessarily bounded) linear operator such that for each $t \in R$, $e^{itH_1}P_a(H_1)\,\mathscr{D}(H_1) \subset \mathscr{D}(J)$ and $C = H_2J - JH_1$, C^*J and JC^*, whose closures are trace class operators, are densely defined. Then $\Omega_{\pm}(H_2, H_1; J)\phi$ $= \lim_{t \to \pm\infty} e^{itH_2}Je^{-itH_1}\phi$ exist for all $\phi \in \mu(H_1) = \{\theta \in R(P_a(H_1))$: $d\langle E_1(\sigma)\theta, \theta\rangle/d\sigma \in L^\infty(R)\}$, where E_1 is the spectral measure of H_1.*

Theorem 1 generalizes Pearson's theorem. In the setup of Pearson's theorem, J is assumed bounded. Given $\phi \in \mathscr{H}_1$, density of $\mu(H_1)$ in

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$R(P_a(H_1))$ and boundedness of $e^{itH_2}Je^{-itH_1}P_a(H_1)$ implies that for some θ_n in $\mu(H_1)$, $f_n(t) = e^{itH_2}Je^{-itH_1}P_a(H_1)\,\theta_n \to f(t) = e^{itH_2}Je^{-itH_1}P_a(H_1)\phi$ uniformly over t. This permits the interchange of limits in the following arguments based on the conclusion of Theorem 1,

$$\begin{aligned}\lim_{t\to\pm\infty} f(t) &= \lim_{t\to\pm\infty}\ \lim_{n\to\infty} f_n(t)\\ &= \lim_{n\to\infty}\ \lim_{t\to\pm\infty} f_n(t)\\ &= \Omega_{\pm}(H_2, H_1; J)\phi,\end{aligned}$$

which is the conclusion of Pearson's theorem.

Pearson's theorem was used by Howland and Kato [2] to prove the Ismagilove multichannel scattering theorem. Likewise, as an application of Theorem 1, we obtain following:

THEOREM 2. *Let A and B be self-adjoint operators (not necessarily bounded) with the same domain $\mathscr{D}$ in a separable Hilbert space $\mathscr{H}$, such that AB, ABA, BAB are densely defined trace class operators. If $H = A + B$, then the absolutely continuous part of H is unitarily equivalent to the direct sum of the absolutely continuous parts of A and B.*

2. Proof of Theorem 1

Let $w(t) = e^{itH_2}Je^{-itH_1}$; for $a > 0$, $F_a(X) = \int_0^a e^{itH_2}Xe^{-itH_1}\,dt$ (let $X: \mathscr{D}(X) \subset \mathscr{H}_1 \to \mathscr{H}_2$ be linear) and $\phi \in \mu(H_1)$. Then as in [5, p. 25] one can see

$$\begin{aligned}(w(t)^*\, w(s) - e^{iaH_1}w(t)^*\, w(s)\, e^{-iaH_1})\phi &\\ = iF_a(e^{itH_1}J^* e^{-i(t-s)H_2}Ce^{-isH_1})\phi &\\ - iF_a(e^{itH_1}C^* e^{-i(t-s)H_2}Je^{-isH_1})\phi. &\qquad (1)\end{aligned}$$

Now for trace class operator T in $\mathscr{H}_1$, we claim

$$\begin{aligned}&|\langle \phi, F_a(e^{iuH_1}Te^{-iuH_1})\phi\rangle|\\ &\quad \leqslant 2\pi\sqrt{(|T|_1)}\ |||\phi|||\ \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n| \int_u^{\infty} |\langle \phi_n, e^{-ixH_1}\phi\rangle|^2\, dx\right)},\end{aligned}$$

where $T = \sum_{n=1}^{\infty} \sigma_n \langle \phi_n, \cdot\rangle\, \phi_n$ ($\{\phi_n\}$ being an orthonormal basis of $\mathscr{H}_1$; $|\cdot|_1$ is the trace norm and $|||\phi||| = \text{ess.sup}\ |d\langle E(\sigma)\phi, \phi\rangle/d\sigma|$). Indeed,

$$\begin{aligned}
&|\langle \phi, F_a(e^{iuH_1}Te^{-iuH_1})\phi\rangle| \\
&\quad = \left|\sum_{n=1}^{\infty} \sigma_n \int_a^{a+u} \langle e^{-ixH_1}\phi, \phi_n\rangle\langle\phi_n, e^{-ixH_1}\phi\rangle\, dx\right| \\
&\quad \leqslant \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n| \int_{-\infty}^{\infty} |\langle e^{-ixH_1}\phi, \phi_n\rangle|^2\, dx\right)} \\
&\qquad \times \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n| \int_u^{\infty} |\langle\phi_n, e^{-ixH_1}\phi\rangle|^2\, dx\right)} \\
&\quad \leqslant 2\pi\, |||\phi|||\, \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n|\right)}\, \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n| \int_u^{\infty} |\langle\phi_n, e^{-ixH_1}\phi\rangle|^2\, dx\right)}.
\end{aligned}$$

The last inequality follows from [5; Lemma 1, p. 23]. Next since $w(t)-w(s)=i\int_s^t e^{iuH_1}Ce^{-iuH_1}\,du$ and C is compact, $w(t)-w(s)$ is compact. So that

$$\lim_{a\to\infty} (w(t)-w(s))\, e^{-iaH_1} = 0 \tag{2}$$

Now,

$$\begin{aligned}
&\langle w(t)\phi, (w(t)-w(s))\phi\rangle \\
&\quad = \lim_{a\to\infty} [\langle w(t)\phi, (w(t)-w(s))\phi\rangle \\
&\qquad - \langle w(t)\, e^{-iaH_1}\phi, (w(t)-w(s))\, e^{-iaH_1}\phi\rangle] \\
&\quad = \lim_{a\to\infty} [\langle \phi, (w(t)^*\, w(t) - e^{iaH_1}w(t)^*\, w(t)\, e^{-iaH_1})\phi\rangle \\
&\qquad - \langle \phi, w(t)^*\, w(s) - e^{iaH_1}w(t)^*\, w(s)\, e^{-iaH_1})\phi\rangle] \\
&\quad = \lim_{a\to\infty} [\langle \phi, F_a(ie^{itH_1}C^*Je^{-itH_1} - ie^{itH_1}J^*Ce^{-itH_1})\phi\rangle \\
&\qquad - \langle \phi, F_a(ie^{itH_1}C^*e^{-i(t-s)H_2}Je^{-itH_1} - ie^{itH_1}J^*e^{-i(t-s)H_2}Ce^{-itH_1})\phi\rangle].
\end{aligned}$$

(The first equality follows by (2) and the last one follows by (1).) Hence,

$$\begin{aligned}
&|\langle w(t)\phi, (w(t)-w(s))\phi\rangle| \\
&\quad \leqslant \lim_{a\to\infty} [|\langle \phi, F_a(e^{itH_1}C^*Je^{-itH_1})\phi| + |\langle \phi, F_a(e^{itH_1}J^*Ce^{-itH_1})\phi\rangle| \\
&\qquad + |\langle \phi, F_a(e^{itH_1}C^*e^{-i(t-s)H_2}Je^{-itH_1})\phi\rangle| \\
&\qquad + |\langle \phi, F_a(e^{itH_1}J^*e^{-i(t-s)H_2}Ce^{-itH_1})\phi\rangle|] \\
&\quad \leqslant 2\pi(\sqrt{(|C^*J|_1)} + \sqrt{(|J^*C|_1)} + \sqrt{(|C^*J|_1)} + \sqrt{(|J^*C|_1)})\, |||\phi||| \\
&\qquad \times \sqrt{\left(\sum_{n=1}^{\infty} |\sigma_n| \int_{\min(s,t)}^{\infty} |\langle\phi_n, e^{-ixH_1}\phi\rangle|^2\, dx\right)} \to 0 \qquad \text{as} \quad s, t\to\infty.
\end{aligned}$$

Therefore, $\|(w(t)-w(s)\phi\| \to 0$ as $t, s\to\infty$. Hence $\lim_{t\to\pm\infty} w(t)\phi$ exist.

3. Proof of Theorem 2

Lemma 1. *Let H be a self-adjoint operator in a Hilbert space $\mathscr{H}$. Then $H\mu(H)$ is dense in $R(P_a(H))$.*

Proof. Let $\phi \in R(P_a(H))$. Then $\langle E(\sigma)\phi, \phi\rangle$ is absolutely continuous and $f(\sigma) = d\langle E(\sigma)\phi, \phi\rangle/d\sigma$ exists almost everywhere and nonnegative, where E is the spectral measure of H. Let $D_n = \{\sigma : f(\sigma)/|\sigma|^2 > n\}$. Then $\{D_n\}$ is a nonincreasing sequence and $\bigcap D_n = \phi$. Let $h_n(\sigma) = (1 - \chi_{D_n}(\sigma))/\sigma$. Then $\phi \in \mathscr{D}(h_n(H))$; indeed,

$$\int |h_n(\sigma)|^2 \, d\langle E(\sigma)\phi, \phi\rangle = \int (1/|\sigma|^2)(1 - \chi_{D_n}(\sigma)) \, f(\sigma) \, d\sigma.$$

Since $(1/|\sigma|^2)(1 - \chi_{D_n}(\sigma)) \, f(\sigma) \leqslant n$, the last integral exists. Now consider $\phi_n = h_n(H)\phi$. We claim that $\phi_n \in \mu(H)$ and $H\phi_n \to \phi$. Indeed,

$$\begin{aligned}\langle E(\sigma)\,\phi_n, \phi_n\rangle &= \langle h_n(H)\, E(\sigma)\phi, h_n(H)\phi\rangle \\ &= \int_{-\alpha}^{\sigma} |h_n(\sigma')|^2 \, d\langle E(\sigma')\phi, \phi\rangle \\ &= \int_{-\alpha}^{\sigma} ((1 - \chi_{D_n}(\sigma'))/|\sigma'|^2) \, f(\sigma') \, d\sigma'.\end{aligned}$$

Hence $d\langle E(\sigma)\phi_n, \phi_n\rangle/d\sigma \leqslant n$. Thus $\phi_n \in \mu(H)$. Next,

$$\begin{aligned}\|H\phi_n - \phi\|^2 = \int &((\sigma h_n(\sigma))^2 - 2\sigma h_n(\sigma) + 1) \\ &\times d\langle E(\sigma)\phi, \phi\rangle \to 0 \qquad \text{as } n \to \infty\end{aligned}$$

(since $h_n(\sigma)\sigma \leqslant 1$ and $h_n(\sigma) \to 1/\sigma$, the dominated convergence theorem is applied).

Lemma 2. *Let A and B be as in Theorem 2. Then the wave operators $\Omega_+(H, A) = \Omega_+(H, A; I)$ and $\Omega_+(H, B)$ exist and their ranges are orthogonal to each other.*

Proof. Let $J = A$. Then $C = HJ - JA = BA$; C^*J and J^*C are all densely defined trace class operators. Hence by Theorem 1, $\Omega_+(H, A; A)\phi$ exists for all $\phi \in \mu(A)$. Since $A\mu(A)$ is dense in $R(P_a(A))$, $\Omega_+(H, A)$ exists. Similarly on taking $J = B$, one can see the existence of $\Omega_+(H, B)$. Next, for $x \in R(P_a(A))$ and $y \in R(P_a(B))$, put $f = Ax$, $g = By$. Then

$$\langle e^{itH}e^{-itA}f, e^{itH}e^{-itB}g\rangle = \langle e^{-itA}f, e^{-itB}g\rangle = \langle BAe^{-itA}x, e^{-itB}y\rangle \to 0$$

as $t \to \infty$ (since BA is compact). Hence the ranges of $\Omega_+(H, A)$ and $\Omega_+(H, B)$ are orthogonal to each other.

Proof of Theorem 2. For $\phi \in \mu(H)$, by Theorem 1, $\phi_A = \Omega_+(A, H; A)\phi$ exists. So that $\Omega_+(H, A)\phi_A = \Omega_+(H, H; A)\phi$ (by the general version of the chain rule [5, p. 35]). Hence $\Omega_+(H, A)\phi_A + \Omega_+(H, B)\phi_B = \Omega_+(H, H; A + B)\phi = \Omega_+(H, H; H)\phi = HP_a(H)\phi = P_a(H)H\phi$. Since $H\mu(H)$ is dense in $R(P_a(H))$, $R(P_a(H)) \subset \Omega_+(H, A)\mathscr{H} + \Omega_+(H, B)\mathscr{H}$. $\Omega_+(H, A)\mathscr{H} + \Omega_+(H, B)\mathscr{H} \subset R(P_a(H))$ is clear [5, Proposition 1, p. 17]. Thus $R(P_a(H)) = \Omega_+(H, A)\mathscr{H} + \Omega_+(H, B)\mathscr{H}$. Hence by Lemma 2, $R(P_a(H)) = \Omega_+(H, A)\mathscr{H} \oplus \Omega_+(H, B)\mathscr{H}$. Now let $U_1 = \Omega_+(H, A)$: $R(P_a(A)) \to R(\Omega_+(H, A))$ and $U_2 = \Omega_+(H, B)$: $R(P_a(B)) \to R(\Omega_+(H, B))$. Then for absolutely continuous vectors x and y in $\mathscr{D}$ of A and B, respectively, $U_1 Ax = HU_1 x$ and $U_2 By = HU_2 y$, so that

$$
\begin{aligned}
U(A \oplus B)(x, y) &= \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} U_1 Ax & 0 \\ 0 & U_2 By \end{bmatrix} = \begin{bmatrix} HU_1 x & 0 \\ 0 & HU_2 y \end{bmatrix} \\
&= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \cdot \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= HU(x, y), \qquad \text{where } U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix};
\end{aligned}
$$

$H_1 = H \mid_{R(\Omega_+(H, A))}$ and $H_2 = H \mid_{R(\Omega_+(H, B))}$.

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